α-Twisted Algebraic Structures

Abdenacer MAKHLOUF

LMIA Mulhouse, France

USTHB, Alger,
November 22, 2015

q-Deformations : Pioneering works

Examples of deformed Lie algebras coming from twisted discretizations of vector fields.

- Aizawa, N., Sato, H., *q-deformation of the Virasoro algebra with central extension*, Phys. Lett. B **256**, no. 1, (1991).
- Chaichian M., Kulish P. and Lukierski J., *q-deformed Jacobi identity, q-oscillators and q-deformed infinite-dimensional algebras*, Phys. Lett. B **237** , no. 3-4 (1990).
- Daskaloyannis C., *Generalized deformed Virasoro algebras,* Modern Phys. Lett. A **7**(9) (1992) 809–816.
- Liu Ke Qin, *Characterizations of the quantum Witt algebra*, Lett. Math. Phys. **24** , no. 4 (1992).
- Curtright, T. L., Zachos, C. K., *Deforming maps for quantum algebras*, Phys. Lett. B **243**, no. 3 (1990)

Paradigmatic example : Lie algebra $\mathfrak{sl}_2(\mathbb{K})$

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

In terms of first order differential operators acting on some vector space of functions in the variable t:

$$E \mapsto \partial, \qquad H \mapsto -2t\partial, \qquad F \mapsto -t^2\partial.$$

To **quasi-deform** $\mathfrak{sl}_2(\mathbb{K})$ means that we replace ∂ by ∂_σ which is a σ-derivation.

Let $\mathcal{A}$ be a commutative, associative $\mathbb{K}$-algebra with unity 1 and $\sigma : \mathcal{A} \to \mathcal{A}$ an algebra endomorphism.

A **σ-derivation** on $\mathcal{A}$ is a $\mathbb{K}$-linear map $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ such that a σ-Leibniz rule holds:

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b).$$

Paradigmatic example : Lie algebra $\mathfrak{sl}_2(\mathbb{K})$

$$[H,E]=2E, \qquad [H,F]=-2F, \qquad [E,F]=H.$$

In terms of first order differential operators acting on some vector space of functions in the variable t:

$$E \mapsto \partial, \qquad H \mapsto -2t\partial, \qquad F \mapsto -t^2\partial.$$

To **quasi-deform** $\mathfrak{sl}_2(\mathbb{K})$ means that we replace ∂ by ∂_σ which is a σ-derivation.
Let $\mathcal{A}$ be a commutative, associative $\mathbb{K}$-algebra with unity 1 and $\sigma : \mathcal{A} \to \mathcal{A}$ an algebra endomorphism.
A **σ-derivation** on $\mathcal{A}$ is a $\mathbb{K}$-linear map $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ such that a σ-Leibniz rule holds:

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b).$$

Example : Jackson q-derivation operator on $\mathcal{A} = \mathbb{K}[t]$

$$\partial_\sigma \quad : \; P(t) \to (D_q P)(t) = \frac{P(qt) - P(t)}{qt - t},$$

here $\sigma P(t) := P(qt)$.

The operator satisfies

$(D_q(PQ))(t) = (D_q P)(t) Q(t) + P(qt)(D_q Q)(t), \quad \sigma$-Leibniz rule

Assume $\sigma(1) = 1$, $\sigma(t) = qt$, $\partial_\sigma(1) = 0$, $\partial_\sigma(t) = 1$.
Then : $\partial_\sigma(t^2) = \partial_\sigma(t \cdot t) = \sigma(t)\partial_\sigma(t) + \partial_\sigma(t)t = (\sigma(t) + t)\partial_\sigma(t)$.

In general, $\partial_\sigma(t^n) = \{n\}_q t^{n-1}$, where $\{n\}_q = \frac{1-q^n}{1-q}$.

Using the commutator

$$[a\partial_\sigma, b\partial_\sigma]_\sigma = (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a))\partial_\sigma \tag{1}$$

The brackets become

$$[H, F]_\sigma = 2\sigma(t)t\partial_\sigma(t)\partial_\sigma = 2qt^2\partial_\sigma = -2qF$$
$$[H, E]_\sigma = 2\partial_\sigma(t)\partial_\sigma = 2E$$
$$[E, F]_\sigma = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma = -(q+1)t\partial_\sigma = \frac{1}{2}(1+q)H.$$

The new bracket satisfies

$$\circlearrowleft_{a,b,c} [\sigma(a)\cdot\partial_\sigma, [b\cdot\partial_\sigma, c\cdot\partial_\sigma]_\sigma]_\sigma = 0. \tag{2}$$

Using the commutator

$$[a\partial_\sigma, b\partial_\sigma]_\sigma = (\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a))\partial_\sigma \tag{1}$$

The brackets become

$$[H, F]_\sigma = 2\sigma(t)t\partial_\sigma(t)\partial_\sigma = 2qt^2\partial_\sigma = -2qF$$
$$[H, E]_\sigma = 2\partial_\sigma(t)\partial_\sigma = 2E$$
$$[E, F]_\sigma = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma = -(q+1)t\partial_\sigma = \frac{1}{2}(1+q)H.$$

The new bracket satisfies

$$\circlearrowleft_{a,b,c} [\sigma(a) \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma]_\sigma = 0. \tag{2}$$

A second example: quantum version of the Witt algebra

Witt algebra is $W = Der(\mathbb{C}[t, t^{-1}])$ with the bracket

$$[t^{i+1}\partial, t^{j+1}\partial] = t^{i+1}\partial(t^{j+1})\partial - t^{j+1}\partial(t^{i+1})\partial = (j-i)t^{i+j+1}\partial$$

q-deforming W with Jackson derivation leads to : (set $e_i = t^{i+1}\partial$)

$$[e_i, e_j]_\sigma = ((j+1)_q - (i+1)_q)e_{i+j} \quad (i \in \mathbb{Z})$$

that satisfy

$$(2)_{q^i}[e_i, [e_j, e_k]_\sigma]_\sigma + (2)_{q^j}[e_j, [e_k, e_i]_\sigma]_\sigma + (2)_{q^k}[e_k, [e_i, e_j]_\sigma]_\sigma = 0.$$

• *Hartwig, Larsson and Silvestrov, Deformations of Lie algebras using σ-derivations, J. of Algebra* **288** *(2005).*

Definition

A **Hom-Lie algebra** is a triple $(V, [\cdot,\cdot], \alpha)$ satisfying

$$[x,y] = -[y,x] \quad \text{(skewsymmetry)}$$
$$\mathcal{J}_\alpha(x,y,z) := \circlearrowleft_{x,y,z} [\alpha(x),[y,z]] = 0 \quad \text{(Hom-Jacobi condition)}$$

for all $x, y, z \in V$, where $\circlearrowleft_{x,y,z}$ cyclic summation.

Remark

In the original definition, there is one extra condition

$$\alpha[x,y] = [\alpha(x), \alpha(y)].$$

We call such algebra multiplicative Hom-Lie algebras.

(τ, σ)-derivations

• *Elchinger, Lundengard, Makhlouf, Silvestrov, Brackets with (τ, σ)-derivations and (p, q)-deformations of Witt and Virasoro algebras,* Forum Math. 2015.

Let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra. We consider two algebra endomorphisms τ and σ of $\mathcal{A}$.
A (τ, σ)-**derivation** $D : \mathcal{A} \to \mathcal{A}$ is an $\mathbb{K}$-linear map such that the following generalized Leibniz identity

$$D(ab) = D(a)\tau(b) + \sigma(a)D(b) \tag{3}$$

is satisfied, with $a, b \in \mathcal{A}$. The set of all (τ, σ)-derivation on $\mathcal{A}$ is denoted by $\mathcal{D}_{\tau,\sigma}(\mathcal{A})$.

Some (τ, σ)-derivations and their Leibniz rules

Operator D	$D(f(t))$	$D((fg)(t))$
Differentiation	$f'(t)$	$D(f(t))g(t) + f(t)D(g(t))$
Shift S	$f(t+1)$	$f(t+1)D(g(t))$
Shift difference	$f(t+1) - f(t)$	$D(f(t))g(t) + f(t+1)D(g(t))$
q-dilatation T_q	$f(qt)$	$f(qt)D(g(t))$
Jackson q-derivative	$\dfrac{f(t) - f(qt)}{t - qt}$	$D(f(t))g(t) + f(qt)D(g(t))$
Jackson symmetric q-derivative	$\dfrac{f(q^{-1}t) - f(qt)}{q^{-1}t - qt}$	$D(f(t))g(q^{-1}t) + f(qt)D(g(t))$
Jackson (p, q)-derivative	$\dfrac{f(pt) - f(qt)}{pt - qt}$	$D(f(t))g(pt) + f(qt)D(g(t))$
p-dilatation derivative	$f'(pt)$	$D(f(t))g(pt) + f(pt)D(g(t))$

Generalized HLS-Theorem

Let $\mathcal{A}$ be a commutative associative unital algebra and $\tau, \sigma : \mathcal{A} \to \mathcal{A}$ different morphisms with τ invertible. Let Δ be a (τ, σ)-derivation. If the equation

$$(\sigma \circ \tau^{-1})(\mathrm{Ann}(\Delta)) \subseteq \mathrm{Ann}(\Delta)$$

holds, then the map $[\ ,\ ]_{\tau,\sigma} : \mathcal{A} \cdot \Delta \times \mathcal{A} \cdot \Delta \to \mathcal{A} \cdot \Delta$ defined by setting

$$[a \cdot \Delta, b \cdot \Delta]_{\tau,\sigma} =$$
$$((\sigma \circ \tau^{-1})(a) \cdot \Delta) \circ (\tau^{-1}(b) \cdot \tau^{-1} \circ \Delta) - ((\sigma \circ \tau^{-1})(b) \cdot \Delta) \circ (\tau^{-1}(a) \cdot \tau^{-1} \circ \Delta)$$

for $a, b \in \mathcal{A}$ is a well-defined $\mathbb{K}$-algebra bracket on the $\mathbb{K}$-linear space $\mathcal{A} \cdot \Delta$, and it satisfies the following identities for $a, b, c \in \mathcal{A}$:

$$[a \cdot \Delta, b \cdot \Delta]_{\tau,\sigma} = \Big((\sigma \circ \tau^{-1})(a)(\Delta \circ \tau^{-1})(b) - (\sigma \circ \tau^{-1})(b)(\Delta \circ \tau^{-1})(a)\Big) \cdot \Delta$$
$$[b \cdot \Delta, a \cdot \Delta]_{\tau,\sigma} = -[a \cdot \Delta, b \cdot \Delta]_{\tau,\sigma}.$$

Moreover if there exists an element $\delta \in \mathcal{A}$ such that

$$\Delta \circ \tau^{-1} \circ \sigma \circ \tau^{-1} = \delta \cdot (\sigma \circ \tau^{-1} \circ \Delta \circ \tau^{-1}),$$

then we have

$$\circlearrowleft_{a,b,c} \Big([\sigma(\tau^{-1}(a)) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_{\tau,\sigma}]_{\tau,\sigma} + \delta \cdot [a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_{\tau,\sigma}]_{\tau,\sigma} \Big) = 0.$$

We will refer to this equation as quasi-Jacobi identity.

Remark

For $\tau = \mathrm{id}$, we recover Hartwig-Larsson-Silvestrov Theorem.

Moreover if there exists an element $\delta \in \mathcal{A}$ such that

$$\Delta \circ \tau^{-1} \circ \sigma \circ \tau^{-1} = \delta \cdot (\sigma \circ \tau^{-1} \circ \Delta \circ \tau^{-1}),$$

then we have

$$\circlearrowleft_{a,b,c} \left([\sigma(\tau^{-1}(a)) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_{\tau,\sigma}]_{\tau,\sigma} + \delta \cdot [a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_{\tau,\sigma}]_{\tau,\sigma} \right) = 0.$$

We will refer to this equation as quasi-Jacobi identity.

Remark

For $\tau = \mathrm{id}$, *we recover Hartwig-Larsson-Silvestrov Theorem.*

Quasi-Lie algebras (Larsson, Silvestrov, 2006)

A *quasi-Lie algebra* is a tuple $(\mathcal{A}, [\ ,\ ], \alpha, \beta, \omega, \theta)$ where

- $\mathcal{A}$ is a $\mathbb{K}$-algebra,
- $[\ ,\ ] : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a bilinear map called a *product* or a *bracket* on $\mathcal{A}$,
- $\alpha, \beta : \mathcal{A} \to \mathcal{A}$, are $\mathbb{K}$-linear maps,
- $\omega : D_\omega \to \mathcal{L}_{\mathbb{K}}(V)$ and $\theta : D_\theta \to \mathcal{L}_{\mathbb{K}}(V)$ is a map with domain of definition $D_\omega, D_\theta \subseteq \mathcal{A} \otimes \mathcal{A}$,

such that the following conditions hold:

ω-symmetry

$$[x, y] = \omega(x, y)[y, x], \qquad \text{for all } (x, y) \in D_\omega;$$

quasi-Jacobi identity

$$\circlearrowleft_{x,y,z} \Big\{ \theta(z, x)\Big([\alpha(x), [y, z]] + \beta[x, [y, z]]\Big) \Big\} = 0,$$

for all $(z,x),(x,y),(y,z) \in D_\theta$.

This class includes

- Hom-Lie algebras with twisting linear map α by specifying in the definition of quasi-Lie algebras $D_\omega = \mathcal{A} \otimes \mathcal{A}$, $\beta = 0_{\mathcal{A}}$ and $\theta(x, y) = \omega(x, y) = -id_{\mathcal{A}}$ for all $(x, y) \in D_\omega = D_\theta$.
- (Hom-)Lie superalgebras,
- $\mathbb{Z}$-graded (Hom-)Lie algebras,
- colour (Hom-)Lie algebras.

(p, q)-deformations of the Witt algebra

$\mathcal{A} = \mathbb{C}[t, t^{-1}]$, for $p, q \in \mathbb{C}^*$ fixed, with $p \neq q$, set $\tau(t) = pt$, and $\sigma(t) = qt$. Note that it implies that $\tau(f(t)) = f(pt)$ and $\sigma(f(t)) = f(qt)$ for any $f \in \mathcal{A}$.
A general formula for D acting on a monomial is as follows

$$D(t^n) = [n]_{p,q} t^n,$$

where $[n]_{p,q}$ is the (p, q)-number defined
for $n \in Z$ and $p \neq q$ by $[n]_{p,q} \dfrac{p^n - q^n}{p - q}$.
For $n \in \mathbb{N}$ we have

$$[n]_{p,q} = \sum_{k=0}^{n-1} p^{n-1-k} q^k \text{ and } [-n]_{p,q} = -(pq)^{-n}[n]_{p,q} = -\sum_{k=0}^{n-1} p^{-1-k} q^{k-n},$$

which allows to define $[n]_{p,p} = np^{n-1}$ for $n \in Z$ when $q = p$.

Theorem

Let $\mathcal{A} = \mathbb{C}[t, t^{-1}]$, $\tau(t) = pt$, $\sigma(t) = qt$ *morphisms of* $\mathcal{A}$ *and* $D = \dfrac{\tau - \sigma}{p - q}$ *a* (τ, σ)*-derivation of* $\mathcal{D}_{\tau,\sigma}(\mathcal{A})$. *Then the* $\mathbb{C}$*-linear space* $W_{p,q} = \bigoplus_{n \in Z} \mathbb{C} \cdot d_n$, *where* $d_n = -t^n \cdot D$, *can be endowed with a structure of Hom-Lie algebra with the bracket defined on generators by*

$$[d_n, d_m] = q^n p^{-n} d_n \circ \tau^{-1} \circ d_m - q^m p^{-m} d_m \circ \tau^{-1} \circ d_n \qquad (4)$$

with commutation relations $[d_n, d_m] = \left(\frac{[n]}{p^n} - \frac{[m]}{p^m}\right) d_{n+m}$, *and with the twist morphism* $\alpha = id + \overline{\sigma \tau^{-1}} : W_{p,q} \to W_{p,q}$ *defined by* $\alpha(d_n) = (1 + (q/p)^n) d_n$.

Proposition

The Hom-Lie algebra

$$\left(W_{p,q}, \quad [d_n, d_m]_{p,q} = \left(\frac{[n]_{p,q}}{p^n} - \frac{[m]_{p,q}}{p^m}\right) d_{n+m}, \quad \alpha_{p,q}(d_n) = (1 + (q/p)^n) d_n\right)$$

is isomorphic (by a factor p on generators) to

$$\left(W_{q/p}, \quad [d_n, d_m]_{q/p} = \left(\{n\}_{q/p} - \{m\}_{q/p}\right) d_{n+m}, \quad \alpha_{q/p}(d_n) = (1 + (q/p)^n) d_n\right)$$

(p, q)-deformations of the Virasoro algebra

Theorem

Every non-trivial one-dimensional central extension of the Hom-Lie algebra $(\mathcal{D}_{\tau,\sigma}(\mathcal{A}), \alpha)$*, where* $\mathcal{A} = \mathbb{C}[t, t^{-1}]$*, is isomorphic to the Hom-Lie algebra* $\mathrm{Vir}_{p,q} = (\hat{L}, \hat{\alpha})$*, where* $\hat{L}$ *has basis* $\{L_n, n \in Z\} \cup \{\boldsymbol{c}\}$ *and bracket relations*

$$[\boldsymbol{c}, \hat{L}]_{\hat{L}} = 0,$$

$$[L_n, L_m]_{\hat{L}} = \left(\frac{[n]}{p^n} - \frac{[m]}{p^m}\right) L_{n+m} + \delta_{n+m,0} \frac{(q/p)^{-n}}{6(1 + (q/p)^n)} \frac{[n-1]}{p^{n-1}} \frac{[n]}{p^n} \frac{[n+1]}{p^{n+1}} \boldsymbol{c},$$

and $\hat{\alpha} : \hat{L} \to \hat{L}$ *is the endomorphism of* $\hat{L}$ *defined by*

$$\hat{\alpha}(L_n) = (1 + (q/p)^n) L_n, \qquad \hat{\alpha}(\boldsymbol{c}) = \boldsymbol{c}.$$

Hom-associative algebras

• *Makhlouf and Silvestrov, Hom-algebra structures, Journal of Generalized Lie Theory and Applications, vol* **2** *(2)* (2008)

Definition

A **Hom-associative algebra** is a triple (V, μ, α) consisting of a linear space V, a bilinear map $\mu : V \times V \to V$ and a homomorphism $\alpha : V \to V$ satisfying

$$\mathfrak{as}_\alpha(x, y, z) := \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = 0$$

A linear map $\phi\ : V \to V'$ is a morphism of Hom-associative algebras if

$$\mu' \circ (\phi \otimes \phi) = \phi \circ \mu \quad \text{and} \qquad \phi \circ \alpha = \alpha' \circ \phi.$$

The algebra is unital if there exists a unit 1 such that

$$\mu(x, 1) = \mu(1, x) = \alpha(x).$$

Functor Hom-Lie

Proposition

To any Hom-associative algebra defined by the multiplication μ and a homomorphism α over a $\mathbb{K}$-linear space V, one may associate a Hom-Lie algebra defined for all $x, y \in V$ by the bracket

$$[x, y] = \mu(x, y) - \mu(y, x)$$

Remark

A construction of the enveloping algebra of Hom-Lie algebra was given by Donald Yau.

Free Hom-associative algebra and enveloping algebra

• D. Yau, Enveloping algebras of Hom-Lie algebras, J. Gen. Lie Theory Appl 2 (2008).
Let (V, μ, α) be a Hom-Nonassociative algebra (Hom-algebra) that is V is a module, μ a multiplication and α a linear map.
The products are defined using the set of weighted trees (T_n^{wt}).
Consider the map

$$\mathbb{K}[T_n^{wt}] \otimes V^{\otimes n} \longrightarrow V$$
$$(\tau; x_1, \cdots, x_n) \longrightarrow (x_1, \cdots, x_n)_\tau$$

inductively via the rules

1. $(x)_i = x$ for $x \in V$, where i denote the 1-tree,
2. If $\tau = (\tau_1 \vee \tau_2)[r]$ then
$(x_1, \cdots, x_n)_\tau = \alpha^r((x_1, \cdots, x_p)_{\tau_1}(x_{p+1}, \cdots, x_{p+q})_{\tau_2})$.

The free Hom-Nonassociative algebra is

$$F_{HNAs}(V) = \oplus_{n\geq 1} \oplus_{\tau \in T_n^{wt}} V_\tau^{\otimes n}$$

where $V_\tau^{\otimes n}$ is a copy of $V^{\otimes n}$.
The multiplication μ_F is defined by

$$\mu_F((x_1, \cdots, x_n)_\tau, (x_{n+1}, \cdots, x_{n+m})_\sigma) = (x_1, \cdots, x_{n+m})_{\tau\vee\sigma}$$

and the linear map is defined by the rule

1. $\alpha_{F|V} = \alpha_V$
2. $\alpha_F((x_1, \cdots, x_n)_\tau) = (x_1, \cdots, x_n)_{\tau[1]}$.

Consider two-sided ideals $I^1 \subset I^2 \subset \cdots I^\infty \subset F_{HNAs(V)}$ where

$$I^1 =< Im(\mu_F \circ (\mu_F \otimes \alpha_F - \alpha_F \otimes \mu_F)) >$$

and $I^{n+1} =< I^n \bigcup \alpha(I^n) >$, $I^\infty = \bigcup_{n\geq 1} I^n$.
The quotient module

$$F_{HAs}(V) = F_{HNAs(V)}/I^\infty.$$

equipped with μ_F and α_F is the free Hom-associative algebra.

The **enveloping Hom-Lie algebra** is obtained by considering the two-sided ideals J^k where

$$J^1 =< Im(\mu_F \circ (\mu_F \circ \alpha - \alpha \circ \mu)); [x,y] - (xy - yx) \text{ for } x, y \in V > .$$

Consider two-sided ideals $I^1 \subset I^2 \subset \cdots I^\infty \subset F_{HNAs(V)}$ where

$$I^1 =< Im(\mu_F \circ (\mu_F \otimes \alpha_F - \alpha_F \otimes \mu_F)) >$$

and $I^{n+1} =< I^n \bigcup \alpha(I^n) >$, $I^\infty = \bigcup_{n\geq 1} I^n$.
The quotient module

$$F_{HAs}(V) = F_{HNAs(V)}/I^\infty.$$

equipped with μ_F and α_F is the free Hom-associative algebra.

The **enveloping Hom-Lie algebra** is obtained by considering the two-sided ideals J^k where

$$J^1 =< Im(\mu_F \circ (\mu_F \circ \alpha - \alpha \circ \mu)); [x,y] - (xy - yx) \text{ for } x, y \in V > .$$

More Generalizations

- Hom-Lie-Admissible algebras, superalgebras
- Hom-Malcev, Hom-alternative, and Hom-Jordan algebras
- *n*-ary Hom-algebras
- Hom-coalgebras, Hom-bialgebras, Hom-Hopf algebras, HYBE, HQYBE, ...
- Rota-Baxter, Hom-dendriform,....
- Hom-Algebroids

Authors: Ammar, Arnlind, Ataguema, Bai, Benayadi, Caenepeel, Calderon-Martin, Casas, Elhamdadi, Ejbehi, Fregier, Gohr, Goyvaerts, Issa, Hartwig, Hou, Laurent-Gengoux, Larsson, Ma, Mabrouk, Makhlouf, Panaite, Silvestrov, Teles, Sheng, Yuan, Yan, Yau, Zhang....

Hom-Lie-Admissible algebras

Let G be a subgroup of the permutations group $\mathcal{S}_3$, a Hom-algebra on (V, μ, α) is said G-Hom-associative if

$$\sum_{\sigma \in G} sgn(\sigma)\mathfrak{as}_\alpha \circ \sigma = 0$$

where $\mathfrak{as}_\alpha \circ \sigma(x_1, x_2, x_3) =$
$\mu(\alpha(x_{\sigma(1)}), \mu(x_{\sigma(2)}, x_{\sigma(3)})) - \mu(\mu(x_{\sigma(1)}, x_{\sigma(2)}), \alpha(x_{\sigma(3)}))$,

Example : G_3-Hom-ass. alg. are Hom-pre-Lie algebras

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(x), \mu(z, y)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y))$$

Proposition

Let G be a subgroup of the permutations group $\mathcal{S}_3$. Then any G-Hom-associative algebra is a Hom-Lie-Admissible algebra.

Hom-alternative, Hom-Malcev algebras

left Hom-alternative identity (reap. right Hom-alternative identity), $\mathfrak{as}_\alpha(x,x,y)=0$, (*resp.* $\mathfrak{as}_\alpha(x,y,y)=0$).
A **Hom-alternative** algebra is one which is both left and right Hom-alternative algebra.
Hom-Malcev identity

$$\mathcal{J}_\alpha(\alpha(x),\alpha(y),[x,z])=[\mathcal{J}_\alpha(x,y,z),\alpha^2(x)].$$

Hom-Alternative algebras are Hom-Malcev-admissible, that is the commutator Hom-algebra of a Hom-alternative algebra defines a Hom-Malcev algebra.

Hom-alternative, Hom-Malcev algebras

left Hom-alternative identity (reap. right Hom-alternative identity), $\mathfrak{as}_\alpha(x,x,y)=0$, (*resp.* $\mathfrak{as}_\alpha(x,y,y)=0$).
A **Hom-alternative** algebra is one which is both left and right Hom-alternative algebra.
Hom-Malcev identity

$$\mathcal{J}_\alpha(\alpha(x),\alpha(y),[x,z])=[\mathcal{J}_\alpha(x,y,z),\alpha^2(x)].$$

Hom-Alternative algebras are Hom-Malcev-admissible, that is the commutator Hom-algebra of a Hom-alternative algebra defines a Hom-Malcev algebra.

Hom-Jordan identity

$$\mathfrak{as}_\alpha(\mu(x,x),\alpha(y),\alpha(x)) = 0$$

Let $(A,\cdot,\alpha)$ be a Hom-alternative algebra. Its plus Hom-algebra $(A,\circ,\alpha)$, where

$$x \circ y = \frac{1}{2}(x \cdot y + y \cdot x),$$

is a Hom-Jordan algebra.

Construction: Twisting principle or Composition method

Theorem

The category of Hom-algebras of a given type is closed under weak morphisms.

Corollary

Let (V, μ) be a classical algebra of a given type and let $\alpha : V \to V$ be an algebra endomorphism. Then (V, μ_α, α), where $\mu_\alpha = \alpha \circ \mu$, is a Hom-algebra of the same type.

Corollary

Let (V, μ, α) be a Hom-algebra of a given type and let $\alpha : V \to V$ be a weak algebra endomorphism. Then $(V, \alpha^n \circ \mu_\alpha, \alpha^{n+1})$ are Hom-algebras of the same type.

Structure of module over Hom-associative algebras.
Let $\mathcal{A} = (V, \mu, \alpha)$ be a Hom-associative $\mathbb{K}$-algebra. An $\mathcal{A}$-module (left) is a triple (M, f, γ) where M is $\mathbb{K}$-vector space and f, γ are $\mathbb{K}$-linear maps, $f : M \to M$ and $\gamma : V \otimes M \to M$, such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes V \otimes M & \stackrel{\mu \otimes f}{\longrightarrow} & V \otimes M \\ \downarrow{\scriptstyle \alpha\otimes\gamma} & & \downarrow{\scriptstyle \gamma} \\ V \otimes M & \stackrel{\gamma}{\longrightarrow} & M \end{array}$$

Remark

A Hom-associative $\mathbb{K}$-algebra $\mathcal{A} = (V, \mu, \alpha)$ is a left $\mathcal{A}$-module with $M = V$, $f = \alpha$ and $\gamma = \mu$.

Superization : Hom-superalgebras, A q-deformed Witt superalgebras

- Ammar and Makhlouf, *Hom-Lie Superalgebras and Hom-Lie admissible Superalgebras*, Journal of Algebra, Volume **324**, Issue 7, (2010)

Definition

A **Hom-Lie superalgebra** is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a superspace $V = V_0 \oplus V_1$, a bilinear map $[\cdot,\cdot] : V \times V \to V$ and an even superspace homomorphism $\alpha : V \to V$ satisfying

$$[x, y] = -(-1)^{|x||y|}[y, x] \tag{5}$$

$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||x|}[\alpha(y), [z, x]] = 0 \tag{6}$$

for all homogeneous element x, y, z in V.

A **Hom-associative superalgebra** is defined similarly.

By composition method we deform the orthosymplectic Lie superalgebra into a Hom-Lie superalgebra.

Example

Let $osp(1,2) = V_0 \oplus V_1$ be the orthosymplectic Lie superalgebra where V_0 is generated by H, X, Y and V_1 is generated by F, G and where the defining relations (we give only the ones with non zero values in the right hand side) are

$$[H,X] = 2X,\ [H,Y] = -2Y,\ [X,Y] = H,$$

$$[Y,G] = F,\ [X,F] = G,\ [H,F] = -F,\ [H,G] = G,$$

$$[G,F] = H,\ [G,G] = -2X,\ [F,F] = 2Y.$$

Let $\lambda \in \mathbb{R}^*$, we consider the linear map $\alpha_\lambda : osp(1,2) \to osp(1,2)$ defined by:

$$\alpha_\lambda(X) = \lambda^2 X,\ \ \alpha_\lambda(Y) = \frac{1}{\lambda^2} Y,\ \ \alpha_\lambda(H) = H,\ \alpha_\lambda(F) = \frac{1}{\lambda} F,\ \ \alpha_\lambda(G) = \lambda G.$$

We provide a family of Hom-Lie superalgebras $osp(1,2)_\lambda = (osp(1,2), [\cdot,\cdot]_{\alpha_\lambda}, \alpha_\lambda)$ where the Hom-Lie superalgebra bracket $[\cdot,\cdot]_{\alpha_\lambda}$ on the basis elements is given, for $\lambda \neq 0$, by:

$$[H,X]_{\alpha_\lambda} = 2\lambda^2 X,\ \ [H,Y]_{\alpha_\lambda} = -\frac{2}{\lambda^2} Y,\ \ [X,Y]_{\alpha_\lambda} = H,$$

$$[Y,G]_{\alpha_\lambda} = \frac{1}{\lambda} F,\ \ [X,F]_{\alpha_\lambda} = \lambda G,\ \ [H,F]_{\alpha_\lambda} = -\frac{1}{\lambda} F,\ \ [H,G]_{\alpha_\lambda} = \lambda G,$$

$$[G,F]_{\alpha_\lambda} = H,\ \ [G,G]_{\alpha_\lambda} = -2\lambda^2 X,\ \ [F,F]_{\alpha_\lambda} = \frac{2}{\lambda^2} Y.$$

$\mathbb{Z}_2$-Graded Hartwig-Larsson-Silvestrov Theorem

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be an associative superalgebra. We assume that $\mathcal{A}$ is supercommutative, that is for homogeneous elements a, b, the identity

$$ab = (-1)^{|a||b|} ba$$

holds.

For example, $\mathcal{A}_0 = \mathbb{C}[t, t^{-1}]$ and $\mathcal{A}_1 = \theta \mathcal{A}_0$ where θ is the Grassman variable ($\theta^2 = 0$). Let σ be an even superalgebra endomorphism of $\mathcal{A}$. Then, $\mathcal{A}$ is a bimodule over itself, the left (resp. right) action is defined by $a \cdot_l b = \sigma(a) b$ (resp. $b \cdot_r a = ba$).

Let $i \in \{0,1\}$. A σ-**derivation** D_i on $\mathcal{A}$ is an endomorphism satisfying:

$$D_i(ab) = D_i(a)b + (-1)^{i|a|}\sigma(a)D_i(b)$$

where $a, b \in \mathcal{A}$ are homogeneous element and $|a|$ is the parity of a. A σ-derivation D_0 is called even $\sigma-$derivation and D_1 is called odd σ-derivation. The set of all σ-derivations is denoted by $Der_\sigma(\mathcal{A})$. $Der_\sigma(\mathcal{A}) = Der_\sigma(\mathcal{A})_0 \oplus Der_\sigma(\mathcal{A})_1$, where $Der_\sigma(\mathcal{A})_0$ (resp $Der_\sigma(\mathcal{A})_1$) is the space of even (resp. odd) σ-derivations.

Let $\sigma : \mathcal{A} \to \mathcal{A}$ be a fixed even endomorphism, Δ an even σ-derivation ($\Delta \in Der_\sigma(\mathcal{A})_0$) and δ be an element in $\mathcal{A}$.

Theorem

If $\sigma(Ann\Delta) \subset Ann\Delta$ holds then the map $[-,-]_\sigma : \mathcal{A}\Delta \times \mathcal{A}\Delta \to \mathcal{A}\Delta$ defined by setting

$$[a\cdot\Delta, b\cdot\Delta]_\sigma = (\sigma(a)\cdot\Delta)\circ(b\cdot\Delta) - (-1)^{|a||b|}(\sigma(b)\cdot\Delta)\circ(a\cdot\Delta) \quad \text{for } a, b \in \mathcal{A} \tag{7}$$

where $\circ$ denotes the composition of functions, is a well-defined superalgebra bracket on the superspace $\mathcal{A}\cdot\Delta$ and satisfies the following identities for $a, b, c \in \mathcal{A}$

$$[a\cdot\Delta, b\cdot\Delta]_\sigma = (\sigma(a)\Delta(b) - (-1)^{|a||b|}\sigma(b)\Delta(a))\cdot\Delta \tag{8}$$

$$[a\cdot\Delta, b\cdot\Delta]_\sigma = -(-1)^{|a||b|}[b\cdot\Delta, a\cdot\Delta]_\sigma \tag{9}$$

In addition, if

$$\Delta(\sigma(a)) = \delta\sigma(\Delta(a)) \quad \text{for} \quad a \in \mathcal{A} \tag{10}$$

holds, then

$$\circlearrowleft_{a,b,c} (-1)^{|a||c|}([\sigma(a)\cdot\Delta, [b\cdot\Delta, c\cdot\Delta]_\sigma]_\sigma + \delta[a\cdot\Delta, [b\cdot\Delta, c\cdot\Delta]_\sigma]_\sigma) = 0$$

q-deformed Witt superalgebra

Let $\mathcal{A}$ be the complex superalgebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ where $\mathcal{A}_0 = \mathbb{C}[t, t^{-1}]$, and $\mathcal{A}_1 = \theta\, \mathbb{C}[t, t^{-1}]$, where θ is the Grassman variable ($\theta^2 = 0$).

The generators of $\mathcal{A}$ are of the form t^n and θt^n for $n \in \mathbb{Z}$.

Let σ be the algebra endomorphism on $\mathcal{A}$ defined by

$$\sigma(t^n) = q^n t^n \quad \text{and} \quad \sigma(\theta) = q\theta.$$

Let ∂_t and ∂_θ be two linear maps on $\mathcal{A}$ defined by

$$\partial_t(t^n) = \{n\} t^n, \quad \partial_t(\theta t^n) = \{n\}\theta t^n,$$
$$\partial_\theta(t^n) = 0, \quad \partial_\theta(\theta t^n) = q^n t^n.$$

The linear map $\Delta = \partial_t + \theta\partial_\theta$ on $\mathcal{A}$ is a an even σ-derivation.

$$\Delta(t^n) = \{n\} t^n,$$
$$\Delta(\theta t^n) = \{n+1\}\theta t^n.$$

Thm $\rightsquigarrow$ Hom-Lie superalgebra structure on the superspace $\mathcal{V} = \mathcal{A} \cdot \Delta$.

Let $\mathcal{V} = \mathcal{A} \cdot \Delta$, be a superspace generated by the elements $X_n = t^n \cdot \Delta$ of parity 0 and the elements $G_n = \theta t^n \cdot \Delta$ of parity 1.
Let $[-,-]_\sigma$ be a bracket on the superspace $\mathcal{V}$ defined by

$$[X_n, X_m]_\sigma = (q^n\{m\} - q^m\{n\})X_{n+m}$$
$$[X_n, G_m]_\sigma = (q^n\{m+1\} - q^{m+1}\{n\})G_{n+m}$$

The others brackets are obtained by supersymmetry or are 0.
Let α be an even linear map on $\mathcal{V}$ defined on the generators by

$$\alpha(X_n) = (1+q^n)X_n$$
$$\alpha(G_n) = (1+q^{n+1})G_n$$

The triple $(\mathcal{V}, [-,-]_\sigma, \alpha)$ is a Hom-Lie superalgebra (q-deformed Witt superalgebra).

Dualization: Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras

• *Makhlouf and Silvestrov,*
(1) Hom-Lie admissible Hom-Coalgebras and Hom-Hopf Algebras, In "Generalized Lie Theory in Math., Physics and Beyond", Springer (2008),
(2) Hom-algebras and Hom-coalgebras , Journal of Algebra and Its Applications Vol. **9** *(2010)*

Definition

A *Hom-coalgebra* is a triple (V, Δ, β) where V is a $\mathbb{K}$-vector space and $\Delta : V \to V \otimes V$, $\beta : V \to V$ are linear maps.
A *Hom-coassociative coalgebra* is a Hom-coalgebra satisfying

$$(\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta. \tag{11}$$

A Hom-coassociative coalgebra is said to be *counital* if there exists a map $\varepsilon : V \to \mathbb{K}$ satisfying $(id \otimes \varepsilon) \circ \Delta = \beta$ and $(\varepsilon \otimes id) \circ \Delta = \beta$.

Let $C = (V, \Delta, \beta)$ be a Hom-coassociative coalgebra. A C-**comodule** (right) is a triple (M, g, ρ) where M is a $\mathbb{K}$-vector space and g, ρ are $\mathbb{K}$-linear maps, $g : M \to M$ and $\rho : M \to M \otimes V$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes V \\ \downarrow^{\rho} & & \downarrow^{g \otimes \Delta} \\ M \otimes V & \xrightarrow{\rho \otimes \beta} & M \otimes V \otimes V \end{array}$$

Remark

A Hom-coassociative coalgebra $C = (V, \Delta, \beta)$ is a right C-comodule with $M = V$, $g = \beta$ and $\rho = \Delta$.

Hom-bialgebras

A *Hom-bialgebra* is a 7-uple $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ where

(B1) (V, μ, α, η) is a Hom-associative algebra with unit η.

(B2) $(V, \Delta, \beta, \varepsilon)$ is a Hom-coassociative coalgebra with counit ε.

(B3) The linear maps Δ and ε are compatible with the multiplication μ, that is

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 \qquad \text{where } e_1 = \eta(1) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \varepsilon(e_1) = 1 \\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x)\,\varepsilon(y) \\ \Delta(\alpha(x)) = \sum_{(x)} \alpha(x^{(1)}) \otimes \alpha(x^{(2)}) \\ \varepsilon \circ \alpha(x) = \varepsilon(x) \end{cases}$$

Given a Hom-bialgebra $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, we show that the vector space $Hom(V, V)$ with the multiplication given by the convolution product carries a structure of Hom-associative algebra.

Proposition

Let $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra. Then the algebra $Hom(V, V)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta$$

and the unit being $\eta \circ \epsilon$ is a unital Hom-associative algebra with the homomorphism map defined by $\gamma(f) = \alpha \circ f \circ \beta$.

Hom-Hopf algebras

A Hom-Hopf algebra over a $\mathbb{K}$-vector space V is given by $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$, where $(V, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ is a bialgebra and S is the antipode that is the inverse of the identity over V for the convolution product.

Remark

Two directions where developed

- $\beta = \alpha$ *(Yau, A.M. and collaborators, ...)*
- $\beta = \alpha^{-1}$ *(called Monoidal Hom-Hopf alg.: Caenepeel, Goyvaerts, ...)*

We have the following properties :

- - The antipode S is unique,
- - $S(\eta(1)) = \eta(1)$,
- - $\varepsilon \circ S = \varepsilon$.
- - Let x be a primitive element $(\Delta(x) = \eta(1) \otimes x + x \otimes \eta(1))$, then $\varepsilon(x) = 0$.
- - If x and y are two primitive elements in $\mathcal{H}$. Then we have $\varepsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a primitive element.
- - The set of all primitive elements of $\mathcal{H}$, denoted by $Prim(\mathcal{H})$, has a structure of Hom-Lie algebra.

Theorem

Let $(V, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra and let $\beta : V \to V$ be a bialgebra endomorphism. Then $(V, \beta \circ \mu, \eta, \Delta \circ \beta, \varepsilon, \beta)$ is a Hom-bialgebra.

Example. Consider the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ generated as usual by $E, F, K^{\pm 1}$ satisfying the relations

$$KK^{-1} = \mathbf{1} = K^{-1}K, KE = q^2EK, \ KF = q^{-2}FK,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

and the bialgebra morphism $\alpha_\lambda : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2)$ defined by

$$\alpha_\lambda(E) = \lambda E, \ \alpha_\lambda(F) = \lambda^{-1}F, \ \alpha_\lambda(K^{\pm 1}) = K^{\pm 1}.$$

Then $\mathcal{U}_q(\mathfrak{sl}_2)_\alpha = (\mathcal{U}_q(\mathfrak{sl}_2), \mu_{\alpha_\lambda}, \Delta_{\alpha_\lambda}, \alpha_\lambda)$ is a Hom-bialgebra.

Cohomology, Deformations and Representations

• *Makhlouf and Silvestrov*, Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras, *Forum Math.* **22** (4) (2010)

• *Sheng Y.*, Representations of hom-Lie algebras, *Algebras and Representation Theory* (2011)

• *Ammar, Ejbehi and Makhlouf* , Cohomology and Deformations of Hom-algebras, *J. of Lie Theory* **21** *No. 4* (2011)

• *Benayadi and Makhlouf* , Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms, Journal of Geometry and Physics **76** (2014)

$$C^n_\alpha(\mathcal{A},\mathcal{A}) = \{\varphi : \mathcal{A}^n \to \mathcal{A} \text{ s.t. } \alpha\circ\varphi = \varphi\circ\alpha^{\otimes n}\}$$

For $n \geq 1$, the n-coboundary operator $\delta^n_{Hom} : C^n_\alpha(\mathcal{A},\mathcal{A}) \to C^{n+1}_\alpha(\mathcal{A},\mathcal{A})$ is defined by

$$\delta^n_{Hom}\varphi(x_0,x_1,\dots,x_n) = \mu\big(\alpha^{n-1}(x_0),\varphi(x_1,x_2,\dots,x_n)\big)$$
$$+\sum_{k=1}^{n}(-1)^k\varphi\big(\alpha(x_0),\alpha(x_1),\dots,\alpha(x_{k-2}),\mu(x_{k-1},x_k),\alpha(x_{k+1}),\dots,\alpha(x_n)\big)$$
$$+(-1)^{n+1}\mu\big(\varphi(x_0,\dots,x_{n-1}),\alpha^{n-1}(x_n)\big).$$

Theorem

$(C_\alpha(\mathcal{A},\mathcal{A}) = \oplus_{n\geq 0}C^n_\alpha(\mathcal{A},\mathcal{A}),\delta_{Hom})$ defines a cohomology complex.

The algebra $C_\alpha(\mathcal{A},\mathcal{A})$

If $\varphi \in C^a_\alpha(\mathcal{A},\mathcal{A})$ and $\psi \in C^b_\alpha(\mathcal{A},\mathcal{A})$ where $a \geq 0, b \geq 0$ then we define $j^\alpha_\varphi(\psi) \in C^{a+b+1}_\alpha(\mathcal{A},\mathcal{A})$ by

$$j^\alpha_\varphi(\psi)(x_0,...,x_{a+b}) = \sum_{k=0}^{b}(-1)^{ak}\psi\big(\alpha^a(x_0),...,\alpha^a(x_{k-1}),\varphi(x_k,...,x_{k+a}),$$
$$\alpha^a(x_{a+k+1}),...,\alpha^a(x_{a+b})\big).$$

and the bracket $[\psi,\varphi]^\Delta_\alpha = j^\alpha_\psi(\varphi) - (-1)^{ab} j^\alpha_\varphi(\psi)$ called Gerstenhaber bracket.

Theorem

The pair $\big(C_\alpha(\mathcal{A},\mathcal{A}), [.,.]^\Delta_\alpha\big)$ *is a graded Lie algebra.*

Deformations of Hom-algebras

Definition

A *1-parameter formal Hom-associative (resp. Hom-Lie) deformation* of $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ is given by a $\mathbb{K}[[t]]$-bilinear map $\mu_t : V[[t]] \times V[[t]] \to V[[t]]$ of the form

$$\mu_t = \sum_{i \geq 0} \mu_i t^i,$$

where each μ_i is a $\mathbb{K}$-bilinear map $\mu_i : V \times V \to V$ such that μ_t is Hom-associative (resp. skew-symmetric and satisfies Hom-Jacobi identity).

Example (**Jackson** $\mathfrak{sl}_2(\mathbb{K})$)

$$[x_1, x_2]_t = 2x_2, \quad [x_1, x_3]_t = -2x_3 - 2tx_3, \quad [x_2, x_3]_t = x_1 + \frac{t}{2}x_1.$$

The linear map α_t is defined by

$$\alpha_t(x_1) = x_1, \ \alpha_t(x_2) = \frac{2+t}{2(1+t)}x_2 = x_2 + \sum_{k=0}^{\infty} \frac{(-1)^k}{2} t^k \ x_2,$$

$$\alpha_t(x_3) = x_3 + \frac{t}{2} x_3.$$

Let $\mathcal{A}_t = (V, \mu_t, \alpha_0)$ be a deformation of a Hom-associative algebra $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ where $\mu_t = \sum_{i\geq 0} \mu_i t^i$.

Proposition

μ_1 is an element of $Z^2_{Hom}(\mathcal{A}_0, \mathcal{A}_0)$. The integrability of μ_1 depends only on its cohomology class.

Proposition

There is, over $\mathbb{K}[[t]]/t^2$, a one-to-one correspondence between the elements of $H^2_{Hom}(\mathcal{A}_0, \mathcal{A}_0)$ and the infinitesimal deformation of $\mathcal{A}_0$ defined by $\mu_t(x, y) = \mu_0(x, y) + \mu_1(x, y)t$.

Theorem

There exists an equivalent deformation $\mathcal{A}'_t = (\mathcal{A}[[t]], \mu'_t, \alpha)$, *where* $\mu'_t = \sum_{i\geq 0} t^i \mu'_i$ *such that* $\mu'_1 \in Z^2_{Hom}(\mathcal{A}, \mathcal{A})$ *and* μ'_1 *does not belong to* $B^2_{Hom}(\mathcal{A}, \mathcal{A})$.
Hence, If $H^2_{Hom}(\mathcal{A}, \mathcal{A}) = 0$ *then every formal deformation is equivalent to a trivial deformation.*

Assume that $H^2_{Hom}(\mathcal{A},\mathcal{A}) \neq 0$, then one may obtain nontrivial one-parameter formal deformations.
We consider the problem of extending a one parameter formal deformation of order $k-1$ to a deformation of order k.

Theorem

Let $(\mathcal{A},\mu,\alpha)$ be a Hom-associative algebra and $\mathcal{A}_t = (\mathcal{A}[[t]],\mu_t,\alpha)$ be a one-parameter formal deformation of $\mathcal{A}$ of order $k-1$, where $\mu_t = \sum_{i\geq 0}^{k-1} t^i \mu_i$.
Then $\delta^2\mu_k = \frac{1}{2}\sum_{p+q=k-1,p>0,q>0}[\mu_p,\mu_q]^{\Delta}_{\alpha} \in Z^3_{Hom}(\mathcal{A},\mathcal{A})$.
Therefore the deformation extends to a deformation of order k if and only if the right-hand side is a 3-coboundary.

Definition

A **Hom-Poisson algebra** is a quadruple $(V, \mu, \{\cdot,\cdot\}, \alpha)$ satisfying

1. (V, μ, α) is a commutative Hom-associative algebra,
2. $(V, \{\cdot,\cdot\}, \alpha)$ is a Hom-Lie algebra,
3. $\{\alpha(x), \mu(y,z)\} = \mu(\alpha(y), \{x,z\}) + \mu(\alpha(z), \{x,y\})$.

Theorem

Let $\mathcal{A}_0 = (V, \mu_0, \alpha_0)$ be a commutative Hom-associative algebra and $\mathcal{A}_t = (V, \mu_t, \alpha_0)$, $\mu_t = \sum_{i\geq 0} \mu_i t^i$, be a deformation of $\mathcal{A}_0$. Consider the bracket defined for $x, y \in V$ by $\{x,y\} = \mu_1(x,y) - \mu_1(y,x)$ where μ_1 is the first order element of the deformation μ_t. Then $(V, \mu_0, \{,\}, \alpha_0)$ is a Hom-Poisson algebra.

Cohomology and deformations of Hom-Lie algebras

We obtain similar results for Hom-Lie algebras.

- The p-cochains are given by p-linear alternating maps commuting with the twist.

- We have a coboundary operator of Chevalley-Eilenberg type.

- We define a Nijenhuis-Richardson bracket which endows the space of cochains with a structure of graded Lie algebra.

Derivations.

Let $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra.
Set $\alpha^{-1} = 0$, $\alpha^0 = Id$, $gl(V)$ the linear group of operators on V.
For any $k \geq 1$, we call $D \in gl(V)$ an α^k-**derivation** of the Hom-Lie $\mathfrak{g}$ if $D \circ \alpha = \alpha \circ D$ and

$$D[x,y] = [D(x), \alpha^k(y)] + [\alpha^k(x), D(y)].$$

Denote by $Der_{\alpha^k}(\mathfrak{g})$ the set of α^k-derivations of $\mathfrak{g}$.
Set $Der(\mathfrak{g}) = \bigoplus_{k \geq -1} Der_{\alpha^k}(\mathfrak{g})$.
For any $D \in Der_{\alpha^k}(\mathfrak{g})$ and $D' \in Der_{\alpha^{k'}}(\mathfrak{g})$, where $k + k' \geq -1$, we have $[D, D'] \in Der_{\alpha^{k+k'}}(\mathfrak{g})$.

Proposition

The pair $(Der(\mathfrak{g}), [\cdot,\cdot])$, where the bracket is the usual commutator, defines a Lie algebra.

Example (Inner α^{k+1}-derivation)

For $u \in V$ satisfying $\alpha(u) = u$, we define the map $ad_k(u) \in gl(V)$ by

$$ad_k(u)(y) = [\alpha^k(y), u] \quad \forall y \in V.$$

Then

$ad_k(u)$ is an α^{k+1}-derivation, that we call an inner α^{k+1}-derivation.

Representations

A **representation** of a Hom-Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ on a vector space M with respect to $A \in gl(M)$ is a linear map $\rho_A : V \longrightarrow End(M)$, satisfying for $x, y \in V$ the identities

$$\begin{aligned} \rho_A(\alpha(x)) \circ A &= A \circ \rho_A(x) \\ \rho_A([x,y]) \circ A &= \rho_A(\alpha(x)) \circ \rho_A(y) - \rho_A(\alpha(y)) \circ \rho_A(x) \end{aligned}$$

Two representations (V, ρ_A) and $(V', \rho'_{A'})$ of $\mathfrak{g}$ are said to be isomorphic if there exists a linear map $\phi : V \to V'$ such that

$$\forall x \in \mathfrak{g} \;\; \rho_{A'}(x) \circ \phi = \phi \circ \rho_A(x) \quad \text{and} \quad \phi \circ A = A' \circ \phi.$$

Example (adjoint representation)

$\mathfrak{g}$ represents on itself via the bracket with respect to α.
The α^s-adjoint representation is denoted by ad_s and defined by

$$ad_s(x)(y) = [\alpha^s(x), y], \quad \forall x, y \in V.$$

Remark. The adjoint representation is not unique.

Semidirect product. Given a representation ρ_A of a Hom-Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot], \alpha)$ on the vector space M.
The skewsymmetric bilinear bracket $[\cdot,\cdot]_{\rho_A} : \wedge^2(V \oplus M) \to V \oplus M$ defined by

$$[(x,u),(y,v)]_{\rho_A} = ([x,y], \rho_A(x)(v) - \rho_A(y)(u))$$

and the linear map $\alpha + A : V \oplus M) \to V \oplus M$ defined by

$$(\alpha + A)(x,y) = (\alpha(x), Au)$$

determine a Hom-Lie algebra $(V \oplus M, [\cdot,\cdot]_{\rho_A}, \alpha + A)$ which is called a semidirect product of the Hom-Lie algebra $\mathfrak{g}$ and M.

Let $(\mathfrak{g}, [\ ,\ ], \alpha)$ be a Hom-Lie algebra and (V, ρ_A) be a representation of $\mathfrak{g}$. Let V^* be the dual vector space of V. We define a linear map $\widetilde{\rho}_{\tilde{A}} : \mathfrak{g} \to End(V^*)$ by $\widetilde{\rho}_{\tilde{A}}(x) = -{}^t\rho_A(x)$. Set $\widetilde{A} =^t A$.

Proposition

The pair $(V^*, \widetilde{\rho}_{\tilde{A}})$, where $\widetilde{\rho}_{\tilde{A}} : \mathfrak{g} \to End(V^*)$ is given by $\widetilde{\rho}_{\tilde{A}}(x) = -{}^t\rho_A(x)$, defines a representation of the Hom-Lie algebra $(\mathfrak{g}, [\cdot, \cdot], \alpha)$ if and only if

$$A \circ \rho_A([x, y]) = \rho_A(x)\rho_A(\alpha(y)) - \rho_A(y)\rho_A(\alpha(x)).$$

Further results

- n-ary algebras.
- Hom-Yang-Baxter equation, quasi-triangular bialgebras, Hom-Lie bialgebras .
- Yetter-Drinfeld modules for Hom-bialgebras.
- twisted tensor products and smash products for Hom-associative algebras.